Institute for Empirical Research in Economics
University of Zurich

Working Paper Series
ISSN 1424-0459

Working Paper No. 172

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October 2003

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October 22, 2003

**Abstract**

*We suggest a simple asset market model in which we analyze competitive and strategic behavior simultaneously. If for competitive behavior two-fund separation holds across periods then it also holds for strategic behavior. In this case the relative prices of the assets do not depend on whether agents behave strategically or competitively. Those agents acting strategically will however invest less in the common mutual fund. Constant relative risk aversion and absence of aggregate risk are shown to be two alternative sufficient conditions for two-period fund separation. With derivatives further strategic aspects arise and strategic behavior is distinct from competitive behavior even for those utility functions leading to two-fund separation.*

*Keywords:* strategic behavior, competitive behavior, two-fund-separation, CAPM.
*JEL classification: C72, G11, D83* .

*We like to thank Piero Gottardi, Enrico De Giorgi, and Rüdiger Frey for valuable discussions. Financial support by the national centre of competence in research "Financial Valuation and Risk Management" is gratefully acknowledged. The national centers in research are managed by the Swiss National Science Foundation on behalf of the federal authorities.
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# 1 Introduction

Standard asset pricing models, as for example the Capital Asset Pricing Model, CAPM, are based on the assumption that all market participants take prices as given. These models give a first intuition for the valuation of assets when portfolio considerations and diversification are important. Moreover, these models are general in the sense that they can be applied to an arbitrary number of assets. They can however not cope with important issues that practitioners face when a lack of liquidity of markets and so called ”slippage” of asset prices are a major concern. The market impact of portfolio decisions is clearly taken into account by institutional investors like pension funds that in most markets hold most of the assets. Also, many hedge funds limit their assets under management because running their strategies with too much capital would eliminate the potential gains from their strategies. Moreover, to benefit from portfolio diversification large investors and hedge funds invest on many markets simultaneously. To cope with these issues while keeping the benefits of portfolio diversification, models of simultaneous strategic interaction on a large number of asset markets are needed.

The idea of this paper is to systematically compare price taking and strategic behavior for a simple asset market with simultaneous competition on arbitrary many assets. The price of this generality is that we limit our attention to symmetric information models with a given complete participation on asset markets. Also in our paper initially investors are not endowed with assets so that changing the market price does not change the wealth of the investor. These aspects should be considered once the difference of strategic and competitive behavior has been understood in our more simple setting.

We consider a two period model with a finite number of states in the second period. A finite number of investors are endowed with wealth that can be spent on first period consumption and on a finite number of assets (bonds and shares) delivering state contingent payoffs in the second period. We assume that every investor is small on the market for first period consumption. First period consumption resembles the real GDP of the world. On this market a large number of producers, pure consumers, and investors interact and so we assume that even large investors are small. Warren Buffet and George Soros, for example, are estimated to manage wealth of approximately a couple of billions USD. This is a huge amount as compared to the market capitalization for individual stocks, while it can be neglected relative to the world's GDP. Asset markets can be complete or incomplete. In our model asset payoffs are the only source to finance second period consumption. All consumers split their wealth between first period consumption and a portfolio of assets in order to maximize intertemporal utility. On the asset market we allow for competitively and also for strategically acting investors. In the first case the investors take prices as given while in the second case they take the market impact of their demand into account. One may argue that these different types of behavior can arise if estimating the market impact is costly because it requires data bases and research facilities so that only some investors have a sufficient incentive to consider their market impact. However, these arguments are beyond our model. Throughout the paper we assume that investors have expected utility functions with homogenous beliefs. As it is well known, for example from Magill and Qunizii [12], the CAPM is the special case of our model which is obtained with only competitively acting investors and quadratic von-Neumann-Morgenstern utility functions. With respect to the strategic behavior the model is similar to the famous Shapley-Shubik [17] Market Game. It will turn out that the number of assets obtained by any investor are given by the ratio of the wealth he has "bet" on that asset divided by the total wealth bet on that asset. One important difference to the Shapley-Shubik model will be that in our model assets are in fixed exogenous supply and income does not depend on the market outcome. Hence in contrast to the Shapley-Shubik model we can easily ensure that the budget restrictions hold. We formulate the agents' decisions in terms of budget shares that are required to add up to one. This formulation of the investment problem in terms of wealth shares, the so called "asset allocation", is standard in finance. It allows to discuss investment decisions based on returns, i.e. payoffs per price of assets. Keeping this convention, our results are more easily comparable to the finance literature.

The aim of our paper is to analyze under which conditions and in which respect strategic investment behavior differs from competitive behavior. To start with, we show that, as the number of investors becomes large, strategic behavior tends to competitive behavior. For a general account of this so called Cournotian foundation of competitive equilibria see Mas-Colell [14]. To obtain this result we let the economy grow in a very symmetric way. In each step one additional identical copy, a replica, of the strategic agents is introduced. Ever since Debreu and Scarf [6] such limit results for replica economies are well know in the general equilibrium literature (cf. Hildenbrand and Kirman [9]). Since in our model the supply is exogenous we increase it proportionally to the number of consumers in our economy. Besides this standard result on the convergence of strategic and competitive behavior we give sufficient conditions for finite economies, such that with respect to the asset allocation problem strategic and competitive behavior become identical. We show that if a form of fund separation holds for competitive behavior – that we suggest to call two period fund separation – then strategically acting agents will form the same portfolio of assets as competitive agents do. Both types of behavior do however differ with respect to the amount of wealth invested in the common mutual fund of assets. The strategically acting investors take into account that their demand will let prices move to their disadvantage and hence invest less into assets as compared to investors with identical characteristics that behave competitively. We also give sufficient conditions for two period fund separation. One of them is constant relative risk aversion, CRRA, the other no-aggregate risk, NAR.

The asset pricing implications of two period fund separation are that the ratios of the prices of risky assets do not depend on whether agents behave competitively or strategically. Moreover, for the case of no-aggregate risk the weight of any asset in the mutual fund turns out to be the expected value of its payoff relative to the total payoff of all assets. This coincides with so called log-optimal pricing (cf. Luenberger [11], chapter 15). It is well known that log-optimal pricing is also obtained if all investors – acting competitively – have logarithmic von-Neumann Morgenstern utilities, a special case of CRRA (cf. Kraus and Litzenberger [10]). We show that this is still true if one allows for strategic behavior.

In the case of the CAPM, heterogeneity in market behavior matters if there is aggregate risk. We observe that then strategically and competitively acting agents do choose substantially different portfolios and hence asset prices differ substantially. On the other hand, if the market does not exhibit aggregate risk, both investors, even if they differ in their strategic behavior, choose the same portfolio. Introducing derivatives leads to a new strategic aspect of the model. On changing demand for the underlying asset, agents can change the payoffs of the derivatives that are based on the prices of that underlying. Indeed in this case it turns out that even with logarithmic utility functions equilibria depend substantially on the form of market behavior.

There is an impressive literature on strategic competition in general equilibrium models. This literature has at least the two lines originating in Gabszewicz and Vial [8] and Shapley-Shubik

[17]. For a recent account see the recent special issue of the Journal of Mathematical Economics, Vol. 39, Nos. 5-6, edited by Gaël Giraud. In this respect our contributionis that we highlight the importance of two-fund separation to obtain more specific results. The cases under which we show that two-period fund separation holds, CRRA and NAR, are clearly not general in the set of all theoretically possible economies but they are important cases studied extensively in the finance literature. Ever since Merton [15] CRRA has become the "work horse" of finance. Also Campbell and Viceira [4] (page 24) argue convincingly that only the case of CRRA is compatible with observed aggregate time series of consumption and risk premia: Wealth has grown considerably while the risk premium remained quite stable over time. The second case for which we can show two-period fund separation is the case of no aggregate risk. Ever since Borch [2] and Malinvaud [13] also this case has been extensively studied in the literature. It is the work horse case for insurance theory.

In the finance literature market impact has been a serious concern, for example, in the field of derivatives (cf. Taleb [19]), when asymmetric information (cf. Brunnermeier [3]) has been considered and in models with endogenous market participation (cf. Pagano [16]). Only the case of derivatives seems sufficiently similar to the model considered here. When presenting our results in section 6.1 concerning derivatives, we will discuss the difference of this literature to our approach.

The rest of the paper is organized as follows. The next section gives the details of the model. Then we suggest an equilibrium concept, that we call Competitive Nash Equilibrium, CNE, in which we study competitive and strategic behavior simultaneously. Having made the equilibrium notion precise we demonstrate the limit theorem. Thereafter, two-period fund separation is defined and it is shown that under standard differentiability assumptions on the utility functions, CNE with two period fund separation do exist. Then we show that CRRA and NAR are sufficient conditions for CNE with two period fund separation. Based on this, the pricing implications are derived. Also, when presenting the general results we give numerical examples for the CAPM case and the log-utility case to illustrate the robustness. Finally, we consider the case of derivatives.

# 2 The model

In the following we define the model we are concerned with. The definition is divided into mainly two parts, the first one concerns the market while the second one concerns the characteristics of the agents on the market.

## 2.1 The market $(\boldsymbol{q}, \boldsymbol{A})$

We consider a 2 periods model with periods $t = 0$ and $t = 1$ of an economy with $S$ states $s$ and $K$ assets $k$. Let us denote by $\mathbb{S}_0 := \{0\} \cup \mathbb{S}$ the set of states, where for convenience $s = 0$ is the state at time 0, and $\mathbb{S} := \{1, .., S\}$ is the set of states at time 1. Let $k = 0$ be the consumption good, while $\mathbb{K} = \{1, .., K\}$ is the set of assets available at time 1.

Let $\mathbf{A} \in \mathbb{R}_+^{K \times S}$ be the matrix of non-negative payoffs of the assets $k \in \mathbb{K}$ over states $s \in \mathbb{S}$. We assume that there are no redundant assets, i.e. $rank\ \boldsymbol{A} = K$. Assets $k \in \mathbb{K}$ are in exogenous supply which is normalized to 1, while the consumption good is in $\infty-$elastic supply. $\boldsymbol{q} \in \mathbb{R}_+^K$ is the price system on the market $\boldsymbol{A}$, while the price for the consumption good is normalized to 1.

## 2.2 The investor $i$

Let $\mathbb{I} = \{1, .., I\}$ be the set of investors on the market. It is assumed that investors have homogenous believes about states in period $\mathbf{1}$, i.e. $\boldsymbol{p}^i = \boldsymbol{p} \in \Delta_+^S$ is the vector of probabilities for states $s \in \mathbb{S}$. An investor is characterized by his first period wealth (endowment) $w^i \in \mathbb{R}_+$ and by his utility $\mathbb{U}^i$ on his consumption in periods $t = 0, 1$. His investment strategy is denoted by $\boldsymbol{\lambda}^i = \boldsymbol{\lambda}^i(w^i) = \left(\lambda_{\mathbf{0}}^i(w^i), \boldsymbol{\lambda}_{\mathbf{1}}^i(w^i)\right) \in \mathbb{R}_+^{K+1}$, where $\lambda_{\mathbf{0}}^i(w^i)$ is his (budget) share of investment in the consumption good and $\boldsymbol{\lambda}_{\mathbf{1}}^i(w^i)$ is his investment in assets $k \in \mathbb{K}$ on $\boldsymbol{A}$. Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^i, i \in \mathbb{I})$ be the vector of investment strategies over the investor population $\mathbb{I}$. Each investor $i$ is supposed to partition all his wealth into 0 period consumption and investment in assets $k \in \mathbb{K}$ to obtain 1-st period consumption. Formally, his budget constraint therefore reads $\sum_{k=0}^{K} \lambda_k^i = 1$ or equivalently $\boldsymbol{\lambda}^i \in \Delta_+^{K+1}$, $\forall i \in \mathbb{I}$. Note that we exclude short sales. This exclusion is a consequence of allowing for strategic behavior. Strategically acting agents know that they could decrease asset prices below zero by going short in assets. As an effect portfolio returns would then become positive and it would pay even more to short the assets further. Without any short sales constraints this would result in unlimited arbitrage opportunities, ruling out the possibility of any type of equilibria.

The consumption of investor $i$ results from his investment strategies as follows. The consumption function of the i-th investor is defined as $\boldsymbol{c}^i : \Delta_+^{K+1} \to \mathbb{R}_+^{S+1}$ by $\boldsymbol{c}^i(\boldsymbol{\lambda}^i) := \Big(c_{\mathbf{0}}^i(\boldsymbol{\lambda}^i), \boldsymbol{c}_{\mathbf{1}}^i(\boldsymbol{\lambda}^i)\Big)$, where $\boldsymbol{c}_{\mathbf{1}}^i(\boldsymbol{\lambda}^i) = \big(\boldsymbol{c}_s^i(\boldsymbol{\lambda}^i), s \in \mathbb{S}\big)$ is the consumption of the $i-th$ investor over states $s \in \mathbb{S}$ according to his investment strategy $\boldsymbol{\lambda}^i$:

$$c_{\mathbf{0}}^i(\boldsymbol{\lambda}^i) = \lambda_{\mathbf{0}}^i \, w^i \tag{1}$$

$$\boldsymbol{c}_s^i(\boldsymbol{\lambda}^i) = \sum_{k \in \mathbb{K}} A_s^k \frac{\lambda_k^i w^i}{q_k} \qquad s \in \mathbb{S}. \tag{2}$$

Recall that all assets are in unit supply. The equilibrium price system $\boldsymbol{q}$ then is given by the investment strategies by requiring $q_k = \sum_{i \in \mathbb{I}} \lambda_k^i w^i$ for all assets $k \in \mathbb{K}$. Hence market clearing prices are the wealth average of the investor‘s strategies.

Given the probabilities $\boldsymbol{p}$, the preferences of the i-th investor are represented by an expected utility function $\mathbb{U}^i : \mathbb{R}_+^{S+1} \to \mathbb{R}$ defined by $\mathbb{U}^i(\boldsymbol{c}^i(\boldsymbol{\lambda}^i)) = u_{\mathbf{0}}^i(c_{\mathbf{0}}^i(\boldsymbol{\lambda}^i)) + \beta^i U_{\mathbf{1}}^i(\boldsymbol{c}_{\mathbf{1}}^i(\boldsymbol{\lambda}^i))$, where $\beta^i$ is a real-valued discount factor, $0 \le \beta^i \le 1$, and $u_{\mathbf{0}}^i : \mathbb{R}_+ \to \mathbb{R}$, and $U_{\mathbf{1}}^i : \mathbb{R}_+^S \to \mathbb{R}$ is defined by

$$U_{\mathbf{1}}^i(\boldsymbol{c}_{\mathbf{1}}^i(\boldsymbol{\lambda}^i)) := \sum_{s \in \mathbb{S}} p_s \, u_{\mathbf{1}}^i(\boldsymbol{c}_s^i(\boldsymbol{\lambda}^i)),$$

where $u_{\mathbf{1}}^i : \mathbb{R}_+ \to \mathbb{R}$. Note that $\mathbb{U}^i = (u_{\mathbf{0}}^i, U_{\mathbf{1}}^i)$. We arrive at

$$[\mathbb{U}^i \circ \boldsymbol{c}^i](\boldsymbol{\lambda}^i) = \mathbb{U}^i(\boldsymbol{c}^i(\boldsymbol{\lambda}^i)) = u_{\mathbf{0}}^i(c_{\mathbf{0}}^i(\boldsymbol{\lambda}^i)) + \beta^i \sum_{s \in \mathbb{S}} p_s u_{\mathbf{1}}^i(\boldsymbol{c}_s^i(\boldsymbol{\lambda}^i)). \tag{3}$$

We make the following standard assumption about the utility function for any $i \in \mathbb{I}$:

- $u_{\boldsymbol{t}}^i : \mathbb{R}_+ \to \mathbb{R}$, $\boldsymbol{t} = 0, 1$, is twice continuously differentiable,
- strictly increasing, strictly concave and
- *(INADA)*: for any $c \in \mathbb{R}_+$, $\frac{\partial}{\partial c} u_{\boldsymbol{t}}^i(c) \to \infty$ as $c \to 0$.

Recently, Alos-Ferrer and Ania [1] have studied Nash equilibria in a similar model when agents are risk neutral. This case requires different techniques. It turns out that all agents choosing a portfolio with weights equal to the relative expected payoffs is the unique Nash equilibrium.

# 3 The equilibrium concept: A first definition

In a competitive equilibrium the agents take the market's price system $\boldsymbol{q}^*$ as given. This situation is different in the Nash equilibrium where investors anticipate that trading alters prices on the market. Investor $j$, thinking strategically, knows that $\tilde{q}_k(\tilde{\boldsymbol{\lambda}}^i(w^i)) = \tilde{\lambda}^i_k w^i + \sum_{j\neq i} \tilde{\lambda}^j_k w^j,\ k \in \mathbb{K}$. Hence for a given wealth distribution, the equilibrium price system $\tilde{\boldsymbol{q}}$ is anticipated to depend on the set of investment strategies $\tilde{\boldsymbol{\lambda}}$, i.e. $\tilde{\boldsymbol{q}} = \tilde{\boldsymbol{q}}(\tilde{\boldsymbol{\lambda}})$. Consequently, any individual's optimal strategy $\tilde{\boldsymbol{\lambda}}^i$ depends directly on the strategies of all other traders $i' \in \mathbb{I}^{(-i)}$. On a market both types of investors, i.e. those following the competitive equilibrium concept and those following the Nash equilibrium concept, coexist.

The consumption of a competitively behaving investor on $\boldsymbol{A}$ therefore is

$$\boldsymbol{c}^i(\boldsymbol{\lambda}^i;\boldsymbol{q}) = \left(\lambda^i_0 w^i, \left(\sum_{k\in\mathbb{K}} A^k_s \frac{\lambda^i_k w^i}{q_k}\right)_{s\in\mathbb{S}}\right), \qquad \boldsymbol{q} \text{ given}, \quad i \in \mathbb{I}_C,$$

while the consumption of a strategically behaving investor $i$ on $\boldsymbol{A}$ relative to investors $\{j \neq i\}$ yields

$$\boldsymbol{c}^i(\boldsymbol{\lambda}^i;\boldsymbol{\lambda}^{(-i)}) = \left(\lambda^i_0 w^i, \left(\sum_{k\in\mathbb{K}} A^k_s \frac{\lambda^i_k w^i}{\lambda^i_k w^i + \sum_{j\neq i}\lambda^j_k w^j}\right)_{s\in\mathbb{S}}\right), \qquad \boldsymbol{\lambda}^{(-i)} = (\lambda^j_k)_{j\neq i} \text{ given} \quad i \in \mathbb{I}_N$$

Note that we have partitioned the set of investors $\mathbb{I}$ into the set of those following the competitive strategy $\mathbb{I}_C$ and those following the Nash strategy, $\mathbb{I}_N$, i.e. $\mathbb{I} = \mathbb{I}_C \cup \mathbb{I}_N$.

Now we are in a position to define Competitive Nash Equilibria:

**Definition 1 (Competitive Nash Equilibrium (CNE))** *Given an economy with wealth distribution $\boldsymbol{w} \in \mathbb{R}^I_{++}$, a Competitive Nash Equilibrium is a pair $(\hat{\boldsymbol{q}}, \hat{\boldsymbol{\lambda}})$, $\hat{\boldsymbol{\lambda}} = (\hat{\boldsymbol{\lambda}}^i, i \in \mathbb{I})$, such that for all investors $i \in \mathbb{I}_C \cup \mathbb{I}_N$ the following conditions are fulfilled simultaneously*

- $$\hat{\boldsymbol{\lambda}}^i \in \operatorname*{argmax}_{\boldsymbol{\lambda}^i\in\Delta^{K+1}_+} \left[\mathbb{U}^i \circ \boldsymbol{c}^i\right](\boldsymbol{\lambda}^i) \tag{4}$$
- $$\hat{q}_k = \sum_{i\in\mathbb{I}} \hat{\lambda}^i_k w^i, \quad k \in \mathbb{K}, \tag{5}$$

*where the consumption of a competitively behaving investor is*

$$\boldsymbol{c}^i(\boldsymbol{\lambda}^i;\hat{\boldsymbol{q}}) = \left(\lambda_0^i w^i, \left(\sum_{k\in\mathbb{K}} A_s^k \frac{\lambda_k^i w^i}{\hat{q}_k}\right)_{s\in\mathbb{S}}\right), \qquad \hat{\boldsymbol{q}} \text{ given}, \quad i \in \mathbb{I}_C,$$

*while the consumption of a strategically behaving investor $i$ relative to investors $\{j \neq i\}$ is*

$$\boldsymbol{c}^i(\boldsymbol{\lambda}^i;\hat{\boldsymbol{\lambda}}^{(-i)}) = \left(\lambda_0^i w^i, \left(\sum_{k\in\mathbb{K}} A_s^k \frac{\lambda_k^i w^i}{\lambda_k^i w^i + \sum_{j\neq i}\hat{\lambda}_k^j w^j}\right)_{s\in\mathbb{S}}\right), \qquad (\hat{\boldsymbol{\lambda}})_{j\neq i} \text{ given} \quad i \in \mathbb{I}_N$$

## 3.1 The FOCs and State price Vectors in CE and NE

In the following we will show that under the conditions made for the utility function the First Order Condition (FOC) is sufficient for determining the optimum. Let us therefore first derive the First Order Condition for CNE.

**Lemma 1** *Consider $i \in \mathbb{I}$ with wealth $w^i \in \mathbb{R}_+$. Defining the scaled nabla operator $\bar{\boldsymbol{\nabla}}^i = (\bar{\nabla}_s^i)_{s\in\mathbb{S}}$, where $\bar{\nabla}_s^i := \beta^i \left(\frac{\partial u_0^i(c_{\boldsymbol{0}}^i)}{\partial c_{\boldsymbol{0}}^i}\right)^{-1} \cdot \frac{\partial}{\partial \boldsymbol{c}_s^i}$, the first order condition for the optimization problem for a CNE $(\boldsymbol{q},\boldsymbol{\lambda})$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^i)$ reads*

$$\boldsymbol{q} \quad = \quad \boldsymbol{A}\, \bar{\boldsymbol{\nabla}}^i U_{\boldsymbol{1}}^i \left(\boldsymbol{c}_{\boldsymbol{1}}^i(\boldsymbol{\lambda}^i)\right) \bullet \mathcal{N}^i(\boldsymbol{\lambda}), \tag{6}$$

*where $\bullet$ denotes the componentwise multiplication of two vectors. $\mathcal{N}^i(\boldsymbol{\lambda})$ has components*

$$\mathcal{N}_k^i(\boldsymbol{\lambda}) = \begin{cases} 1 & i \in \mathbb{I}_C \\ 1 - \frac{\lambda_k^i w^i}{\sum_j \lambda_k^j w_j} & i \in \mathbb{I}_N \end{cases} \tag{7}$$

*Furthermore, the First Order Condition is necessary and also sufficient for determining the maximum.*

**Proof 1** *The agent's optimization problem reads* $\max[\mathbb{U}^i \circ \boldsymbol{c}^i](\boldsymbol{\lambda}^i)$ *subject to the conditions $\sum_{k=0}^K \lambda_k^i = 1$ and $\lambda_k^i \geq 0$. Defining $g(\boldsymbol{\lambda}^i) := \sum_{k'=0}^K \lambda_{k'}^i$, the first order conditions (FOCs) are*

$$\frac{\partial}{\partial_{\lambda_k^i}} \left[\mathbb{U}^i \circ \boldsymbol{c}^i\right](\boldsymbol{\lambda}^i) \leq \alpha \, \frac{\partial}{\partial_{\lambda_k^i}} g(\boldsymbol{\lambda}^i) + \sum_{k=0}^K \alpha_k', \qquad \mathbb{R} \ni \alpha, \alpha_k' \geq 0 \; \forall k \in \mathbb{K}$$

*Because of the INADA assumption about the utility function $\mathbb{U}^i$, we can exclude the cases $\{\alpha = 0\} \vee \{\alpha_k' = 0\}_{k=0..K}$ and hence all solutions are interior. Since $\mathbb{U}^i$ is assumed to be increasing, the FOCs hold with equality and we obtain*

$$w^i = \beta^i \sum_s p_s \frac{\frac{\partial}{\partial \boldsymbol{c}_s^i} u_{\boldsymbol{1}}^i(\boldsymbol{c}_s^i(\boldsymbol{\lambda}^i))}{\frac{\partial}{\partial c_{\boldsymbol{0}}^i} u_{\boldsymbol{0}}^i(c_{\boldsymbol{0}}^i(\boldsymbol{\lambda}^i))} \left(\frac{\partial \boldsymbol{c}_s^i(\boldsymbol{\lambda}^i)}{\partial \lambda_k^i}\right). \tag{8}$$

*Denoting by* $\bar{\nabla}^i_s$ *the operator for the scaled partial derivative*

$$\bar{\nabla}^i_s := \beta^i \left(\frac{\partial u^i_{\mathbf{0}}(\boldsymbol{c}^i_{\mathbf{0}})}{\partial \boldsymbol{c}^i_{\mathbf{0}}}\right)^{-1} \cdot \frac{\partial}{\partial \boldsymbol{c}^i_s},$$

*the FOC for the k-th component in* $\mathbb{K}$ *becomes*

$$w^i = \sum_s \bar{\nabla}^i_s \, U^i_{\mathbf{1}} \left(\boldsymbol{c}^i_{\mathbf{1}}(\boldsymbol{\lambda}^i)\right) \left(\frac{\partial \boldsymbol{c}^i_s(\boldsymbol{\lambda}^i)}{\partial \lambda^i_k}\right).$$

*A straightforward calculation yields*

$$\begin{aligned}
\frac{\partial \boldsymbol{c}^i_s(\boldsymbol{\lambda}^i)}{\partial \lambda^i_k} &= w^i \sum_{k'} A^{k'}_s \left(\frac{\partial \lambda^i_{k'}}{\partial \lambda^i_k}\frac{1}{q_{k'}} - \frac{\lambda^i_{k'}}{(q_{k'})^2}\frac{\partial q_{k'}}{\partial \lambda^i_k}\right) \\
&= \frac{w^i A^k_s}{q_k}\left(1 - \frac{\lambda^i_k w^i}{\sum_j \lambda^j_k w^j}\delta^i\right),
\end{aligned}$$

*where* $\delta^i = 1$ *if* $i \in \mathbb{I}_N$ *and* $0$ *if* $i \in \mathbb{I}_C$*. Thus, by defining the so called Nash term*

$$\mathcal{N}^i_k(\boldsymbol{\lambda}) = 1 - \frac{\lambda^i_k w^i}{\sum_j \lambda^j_k w^j}\delta^i. \qquad k \in \mathbb{K} \tag{9}$$

*the First Order Condition takes the form*

$$\begin{aligned}
q_k &= \left(\sum_s A^k_s \, \bar{\nabla}^i_s U^i_{\mathbf{1}} \left(\boldsymbol{c}^i_{\mathbf{1}}(\boldsymbol{\lambda}^i)\right)\right)\mathcal{N}^i_k(\boldsymbol{\lambda}) &(10)\\
\boldsymbol{q} &= \boldsymbol{A} \, \bar{\boldsymbol{\nabla}}^i \, U^i_{\mathbf{1}} \left(\boldsymbol{c}^i_{\mathbf{1}}(\boldsymbol{\lambda}^i)\right) \bullet \mathcal{N}^i(\boldsymbol{\lambda}) \qquad i \in \mathbb{I}, &(11)
\end{aligned}$$

*where* $\bar{\boldsymbol{\nabla}}^i$ *is the vector of the scaled partial derivatives* $\bar{\nabla}^i_s$ *defined above and* $\bullet$ *denotes the componentwise multiplication of two vectors.*

*Note that FOC for CE and NE only differ by a factor* $\mathcal{N}^i(\boldsymbol{\lambda})$*. Moreover, note that for* $\delta = 1$ *we get*

$$\mathcal{N}^i_k(\boldsymbol{\lambda}) = \frac{\sum_{j \neq i} \lambda^j_k w^j}{\sum_j \lambda^j_k w^j}. \tag{12}$$

*It remains to be shown that this condition is sufficient for determining the maximum. This follows from above because* $\boldsymbol{c}^i_s$ *is concave in each component* $s$ *since*

$$\left[\frac{\partial^2 \boldsymbol{c}^i_s}{\partial \lambda^i_k \partial \lambda^i_{k'}}\right] = \begin{cases} -\frac{A^k_s w^i}{q^2_k}\delta^i \leq 0 & k = k' \\ 0 & k \neq k' \end{cases}$$

*Hence, as a composition of concave functions,* $[\mathbb{U}^i \circ \boldsymbol{c}^i]$ *is componentwise concave and so the FOC is necessary and also sufficient for determining the maximum.*

In the case of a population with homogenous behavior this reduces to the standard definitions.

**Corollary 1 (Competitive equilibrium)** *Consider a 2 period economy with $I$ investors $\mathbb{I} = \mathbb{I}_C$ and wealth $w \in \mathbb{R}^I_+$, where investor $i$ has an utility function $\mathbb{U}^i = (u^i_{\mathbf{0}}, U^i_{\mathbf{1}}) : \mathbb{R}^{S+1}_+ \to \mathbb{R}$ as defined above. Then a competitive equilibrium is a tuple $(\boldsymbol{q}^*, \boldsymbol{\lambda}^*)$, $\boldsymbol{\lambda}^* = (\boldsymbol{\lambda}^{*i}, i \in \mathbb{I})$, where $\boldsymbol{q}^* \in \mathbb{R}^K_+$ and $\boldsymbol{\lambda}^{*i} \in \Delta^{K+1}_+$ such that*

$$\boldsymbol{q}^* = \boldsymbol{A}\,\bar{\boldsymbol{\nabla}}^i U^i_{\mathbf{1}}(\boldsymbol{c}^i_{\mathbf{1}}(\boldsymbol{\lambda}^{*i})) \qquad \forall i \in \mathbb{I}_C \qquad \textit{where} \tag{13}$$

$$\boldsymbol{c}^i_s(\boldsymbol{\lambda}^{*i}) = \sum_{k\in\mathbb{K}} A^k_s \frac{\lambda^{*i}_k w^i}{\sum_{j\in\mathbb{I}} \lambda^{*j}_k w^j} \tag{14}$$

**Corollary 2 (Nash equilibrium)** *Consider a 2 period economy with $I$ investors $\mathbb{I} = \mathbb{I}_N$ and wealth $w \in \mathbb{R}^I_+$, where investor $i$ has an utility function $\mathbb{U}^i = (u^i_{\mathbf{0}}, U^i_{\mathbf{1}}) := \mathbb{R}^{S+1}_+ \to \mathbb{R}$ as defined above. Then a Nash equilibrium is a pair $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\lambda}})$, $\tilde{\boldsymbol{\lambda}} = (\tilde{\boldsymbol{\lambda}}^i, i \in \mathbb{I})$, where $\tilde{\boldsymbol{q}} \in \mathbb{R}_+$ and $\tilde{\boldsymbol{\lambda}}^i \in \Delta^{K+1}_+$ such that*

$$\tilde{\boldsymbol{q}} = \boldsymbol{A}\,\bar{\boldsymbol{\nabla}}^i U^i_{\mathbf{1}}(\boldsymbol{c}^i_{\mathbf{1}}(\tilde{\boldsymbol{\lambda}^i})) \bullet \mathcal{N}^i(\tilde{\boldsymbol{\lambda}}) \qquad \forall i \in \mathbb{I}_N \qquad \textit{where} \tag{15}$$

$$\boldsymbol{c}^i_s(\tilde{\boldsymbol{\lambda}^i}) = \sum_{k\in\mathbb{K}} A^k_s \frac{\tilde{\lambda}^i_k w^i}{\sum_{j\in\mathbb{I}} \tilde{\lambda}^j_k w^j} \tag{16}$$

# 4 A Limit Theorem

While in general CE and NE differ for small economies, both coincide in the limit of a large economy. Let us consider a market on which a N-multiplicity of investors act, i.e. we have $N \cdot I$ agents. Each agent $i$ is supposed to have $N$ identical replica $i(1), .., i(N)$ having identical utility functions $\mathbb{U}^{i,n} = \mathbb{U}^i$ and income distribution $w^{i,n} = w^i$ following the strategies $\boldsymbol{\lambda}^{i,n}$. We assume that supply or equivalently payoffs are scaled appropriately, i.e. $\mathbf{A}^{(N)} = \left(f_k(N) A^k_s\right)_{k\in\mathbb{K}, s\in\mathbb{S}}$, where $f_k(N) \geq 0$ for all $k \in \mathbb{K}$. Then for strategically acting agents $\mathcal{N}^{i,(N)}_k(\boldsymbol{\lambda}) := 1 - \frac{\lambda^i_k w^i}{N \sum_{j=1}^I \lambda^j_k w^j}$ for $k \in \mathbb{K}$. The following statement follows immediately from Theorem 1.

**Corollary 3** *Let $\tilde{\boldsymbol{\lambda}}^{i,n} \in \Delta^{K+1}_+$, $i = 1..I, n = 1..N$ be a Nash optimal investment strategy for the $i-th$ investor in a $N$ fold replica economy as defined above. Then $\tilde{\boldsymbol{\lambda}}^{(i,n)} \to \boldsymbol{\lambda}^{*,i}$ as $N \to \infty$ provided that $\frac{f_k(N)}{N} \to 1$ for all $k$, where $\boldsymbol{\lambda}^{*i}$ is the optimal competitive strategy of investor $i$ in the one fold replica, $N = 1$.*

**Proof 2** *According to equation 11 the FOC for the N replica NE economy is as follows*

$$\tilde{\boldsymbol{q}} = \boldsymbol{A}^{(N)} \bar{\boldsymbol{\nabla}}^i U^i_{\mathbf{1}}(\boldsymbol{c}^i_{\mathbf{1}}(\tilde{\boldsymbol{\lambda}^i}) \bullet \mathcal{N}^{i,(N)}(\tilde{\boldsymbol{\lambda}}),$$

*where* $\tilde{q}_k = \sum_{i,n=1}^{I,N} \tilde{\lambda}_k^{i,n} w^{i,n} = N \sum_{i=1}^{I} \tilde{\lambda}_k^i w^i$ *such that we have*

$$N \sum_{i=1}^{I} \lambda_k^i w^i = \left( \sum_s A_s^k \, \bar{\nabla}_s^i U_{\mathbf{1}}^i(\boldsymbol{c}_{\mathbf{1}}^i(\tilde{\boldsymbol{\lambda}^i}) \right) f_k(N) \, \mathcal{N}_k^{i,(N)}(\tilde{\boldsymbol{\lambda}})$$

*Finally observe that* $\mathcal{N}_k^{i,(N)}(\tilde{\boldsymbol{\lambda}}) \to 1$ *as* $N \to \infty$*. Hence if* $N \to \infty$ *and* $f_k(N)/N \to 1$ *the expression reduces to the FOC of CE. Therefore, under these conditions* $\frac{\tilde{q}_k}{q_k^*} \to 1$ *and hence the claim follows.*

# 5 Two-Period Fund Separation

In this section we demonstrate that increasing the size of the economy is not the only case in which competitive and strategic behavior become similar. Actually for any finite economy it is shown that for this to hold a form of two-fund separation is decisive. Recall that similar forms of two fund separation are known to be the basis for many important results in finance, as for examples the CAPM. We will discuss the distinction between the two fund separation in our paper and that of CAPM once we have defined our notion. The investment strategy of investor $i$ is $\boldsymbol{\lambda}^i \in \Delta_+^{K+1} \subset \mathbb{R}^K$. We now represent each investment strategy in terms of elementary investment strategies $\boldsymbol{\lambda}_k \in \mathbb{R}_+^K$, where

$$(\boldsymbol{\lambda}_k)_{k'} = \begin{cases} 0 & k' \neq k \\ 1 & k' = k \end{cases} .$$

Hence $\boldsymbol{\lambda}_k$ is the relative investment in the asset $k \in \mathbb{K}$. Each $K$-subset of elementary investment strategies clearly constitutes a basis for the space of investment strategies. Thus each investment strategy $\boldsymbol{\lambda}^i \in \Delta_+^{K+1}$ can be written as a linear superposition of these elementary investment strategies

$$\boldsymbol{\lambda}^i = \sum_{k=0}^{K} \lambda_k^i \boldsymbol{\lambda}_k, \qquad \lambda_k^i \in [0,1], \ \sum_{k=0}^{K} \lambda_k^i = 1$$

— Please insert Figure 1 about here —

Two Fund Separation concerns the partition of an optimal fund into two regimes. Here we consider separation of an equilibrium fund over periods, i.e. the partitioning of wealth distribution

$\boldsymbol{w}$ into 0 period consumption and 1 period portfolio selection on the security market $\boldsymbol{A}$. We therefore call this separation Two-Period Fund Separation ($2pFS$).

**Definition 2 (Two-period-Fund Separation (2pFS))** *Let $\boldsymbol{\lambda}^i(w^i) \in \Delta_+^{K+1}$ be the investment strategy of agent $i$ on the market $\boldsymbol{A}$ in a CNE economy given his wealth $w^i \in \mathbb{R}_+$. Then $2pFS$ holds if and only if for all investment strategies $\boldsymbol{\lambda}^i \in \Delta_+^{K+1}$, there exists a unique common portfolio investment $\bar{\boldsymbol{\lambda}} \in \Delta_+^{K+1}$ for all investors $i$ on the security market $\boldsymbol{A}$ such that*

$$\boldsymbol{\lambda}^i(w^i) \in \left\langle \boldsymbol{\lambda}_0, \bar{\boldsymbol{\lambda}} \right\rangle \cap \Delta_+^{K+1} \tag{17}$$

*for an equilibrium strategy for all investors $i \in \mathbb{I}$.*

Since $dim\left\langle \boldsymbol{\lambda}_0, \bar{\boldsymbol{\lambda}} \right\rangle = 1$, this is equivalent to saying that each investment strategy is uniquely represented by a real number $\lambda_0^i(w^i) \in [0,1]$:

$$\boldsymbol{\lambda}^i(w^i) := \lambda_0^i(w^i)\boldsymbol{\lambda}_0 + (1 - \lambda_0^i(w^i))\bar{\boldsymbol{\lambda}},$$

where $\lambda_0^i(w^i)$ is the relative investment of investor $i$ in 0 period consumption and $\bar{\boldsymbol{\lambda}}$ is the unique mutual fund on $\boldsymbol{A}$. This situation is displayed in Figure 1. In other words, under $2pFS$ optimal investment strategies only differ in relative investments in 0 period consumption. Investment strategies then have the following representation with respect to the coordinate system $(\boldsymbol{\lambda}_0, \bar{\boldsymbol{\lambda}})$:

$$\boldsymbol{\lambda}^i(w^i) = \Big(\lambda_0^i(w^i), (1 - \lambda_0^i(w^i))\Big) \tag{18}$$

Standard two fund separation (Cass and Stiglitz [5]) refers to the separation of investment decisions in a riskless asset and a fund of risky asset components. In our model zero period consumption plays a similar role as the riskless asset in standard two fund separation since it also guarantees risk free payoffs - however delivered one period before the other assets pay off. If in our model some of the assets $k \in \mathbb{K}$ were risk free then, due to borrowing and saving, the different time periods of the riskless payoffs would not matter. Yet our model uses a slightly stronger structure than only separating between riskless and risky payoffs. In our model additive separability over time and the INADA conditions imply that one has to consume something in period 0, i.e. riskfree consumption is essential and cannot be substituted by possibly risky consumption.

The main question is which properties on the market structure $\mathbf{A}$ and on the utility functions $\mathbb{U}^i$ permit 2pFS. Our first statement concerns the market, the second the utility functions. We first show in **Theorem 2** that 2period fund separation holds for any economy provided there is no aggregate risk. Ever since Borch and Malinvaud [2, 13] this case has been intensively studied in the literature. Furthermore, as **Theorem 3** shows, 2pFS also holds if utility functions are CRRA. Cass and Stiglitz [5] have already found the importance of CRRA for fund separation. In our model with only one period, CRRA is equivalent to having a single fund on assets. Let us consider these cases in more detail.

**Theorem 2** *Consider an economy without aggregate risk, i.e.* $\sum_k A_s^k = a$*,* $a \in \mathbb{R}_+$ *and non zero endowment, i.e.* $(w^i)_i \in \mathbb{R}_+^I$*. Then there exists an equilibrium in which 2pFS holds, the mutual fund being*

$$\bar{\boldsymbol{\lambda}}_k = \sum_{s \in \mathbb{S}} p_s \frac{A_s^k}{\sum_{k'} A_s^{k'}} = \frac{1}{a} \sum_{s \in \mathbb{S}} p_s A_s^k.$$

This particular mutual fund preserves a special notation, $\bar{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^*$.

**Proof 3** *Obviously* $\sum_k \bar{\boldsymbol{\lambda}}_k = 1$*. We show that, provided there is no aggregate risk, there exists an* $\lambda_0^i \in [0,1]$ *such that* $\boldsymbol{\lambda}^i = \lambda_0^i \boldsymbol{\lambda}_0 + (1-\lambda_0^i)\bar{\boldsymbol{\lambda}}$ *is an CNE equilibrium. Suppose 2pFS holds. Let* $\hat{\lambda}_0 := \left(\hat{\lambda}_0^1, .., \hat{\lambda}_0^I,\right)$ *and define* $\nu^i(\hat{\lambda}_0) = \frac{1-\hat{\lambda}_0^i}{\sum_j (1-\hat{\lambda}_0^j) w^j}$*, then the Nash terms becomes*

$$\mathcal{N}_k^i(\hat{\lambda}_0) = 1 - \nu^i(\hat{\lambda}_0) w^i \qquad \forall k \in \mathbb{K},$$

*while consumption reduces to* $c_{\mathbf{0}}^i(\hat{\lambda}_0) = \hat{\lambda}_0^i w^i$ *and* $\boldsymbol{c}_s^i(\hat{\lambda}_0) = \left(\sum_k A_s^k\right) \nu^i(\hat{\lambda}_0) w^i$ *for all* $s \in \mathbb{S}$*. Note that if* $\hat{\lambda}_0^i \to 1$*, then* $\nu^i(\hat{\lambda}_0^i) \to 0$ *and so* $\boldsymbol{c}_s^i \to 0$*, while if* $\hat{\lambda}_0^i \to 0$ *these quantities remain finite.*

*If there is no aggregate risk, i.e.* $\sum_k A_s^k = a$*, the consumption is independent of s, i.e.* $\boldsymbol{c}^i(\hat{\lambda}_0) = a\nu^i(\hat{\lambda}_0) w^i\, \mathbf{1}$*, where* $\mathbf{1} = (1, .., 1)$ *is an S-dimensional vector and* $\boldsymbol{c}^i$ *is constant over all states s. By defining* $c^i(\hat{\lambda}_0) = a\nu^i(\hat{\lambda}_0) w^i$*, we write* $\boldsymbol{c}^i(\hat{\lambda}_0) = c^i(\hat{\lambda}_0)\mathbf{1}$*. Under the NAR assumption with these definition the FOC for CNE takes the form*

$$\sum_s \bar{\nabla}_s^i\, U_{\mathbf{1}}^i(c^i(\hat{\lambda}_0)\mathbf{1}) A_s^k \Big(1 - \nu^i(\hat{\lambda}_0) w^i\Big) = \bar{\boldsymbol{\lambda}}_k \sum_j (1-\hat{\lambda}_0^j) w^j$$

*Thus we arrive at*

$$\sum_j (1-\hat{\lambda}_0^j) w^j = \left(\sum_s \frac{A_s^k p_s}{\bar{\boldsymbol{\lambda}}_k}\right) \frac{\beta^i \frac{\partial}{\partial c^i} u_{\mathbf{1}}^i(c^i(\hat{\lambda}_0))}{\frac{\partial}{\partial c_{\mathbf{0}}^i} u_{\mathbf{0}}^i(c_{\mathbf{0}}^i(\hat{\lambda}_0)} \Big(1 - \nu^i(\hat{\lambda}_0) w^i\Big) \tag{19}$$

$$= a\, \frac{\beta^i \frac{\partial}{\partial c^i} u_{\mathbf{1}}^i(c^i(\hat{\lambda}_0))}{\frac{\partial}{\partial c_{\mathbf{0}}^i} u_{\mathbf{0}}^i(c_{\mathbf{0}}^i(\hat{\lambda}_0)} \Big(1 - \nu^i(\hat{\lambda}_0) w^i\Big) \tag{20}$$

*It remains to be shown that a solution in $\hat{\lambda}_0$ exists. Therefore note that the left hand side $\sum_j(1-\hat{\lambda}_0^j)w^j$ is positive and finite for any $\hat{\lambda}_0$. If $\hat{\lambda}_0^i \to 0$ then $0 < \nu^i(\hat{\lambda}_0) < \infty$ and the term $\beta^i \frac{\partial}{\partial_{c^i}} u_{\mathbf{1}}^i(c^i(\hat{\lambda}_0))$ is positive and finite, while $\frac{\partial}{\partial_{c_{\mathbf{0}}^i}} u_{\mathbf{0}}^i(\nu^i(\hat{\lambda}_0)w^i) \to \infty$ and hence the right hand side tends to 0 as $\hat{\lambda}_0^i \to 0$. On the other hand, if $\hat{\lambda}_0^i \to 1$ then $\nu^i(\hat{\lambda}_0) \to 0$ and therefore $c^i(\hat{\lambda}_0) \to 0$. While $0 < \frac{\partial}{\partial_{c_{\mathbf{0}}^i}} u_{\mathbf{0}}^i(\nu^i(\hat{\lambda}_0)) < \infty$, $\frac{\partial}{\partial_{c^i}} u_{\mathbf{1}}^i(c^i(\hat{\lambda}_0)) \to \infty$ and hence the right hand side tends to $\infty$ as $\hat{\lambda}_0^i \to 1$. Since both sides are continuous in $\hat{\lambda}_0$, a solution exists.*

In the mutual fund $\bar{\boldsymbol{\lambda}}$, the weight of any asset turns out to be the expected value of its payoff relative to the total payoff of all assets. This coincides with so called log-optimal pricing (cf. Long [**?**]). Indeed the same mutual fund is obtained in the case of logarithmic utility functions - a special case of CRRA which is covered by our Theorem 3.

— Please insert Figure 2 about here —

Some intuition for this result holding in the case of no aggregate risk is provided by referring to efficient risk sharing (cf. Borch [2] and Malinvaud [13]) as displayed in Figure 2. Since all agents have expected utility functions and beliefs are homogenous, in the case of no aggregate risk efficient risk sharing is obtained at "fair" asset prices, i.e. at prices that are equal the expected payoffs of the assets. In this case every consumer receives a fraction of the aggregate payoffs and hence no individual needs to carry any risk. As Borch and Malinvaud have shown, this is clearly a competitive equilibrium.

When agents take their market impact into account they realize that their budget sets are not given by a budget line but by a curve that lies below the budget line and coincides with it only at the point of efficient risk sharing. This is because any demand different to the efficient level would turn prices to the disadvantage[1] of the agent deviating from the efficient allocation. This intuition can be derived from a reinterpretation of the first-order-condition

$$\boldsymbol{q}_k/\mathcal{N}_k^i(\boldsymbol{\lambda}) = \sum_s A_s^k\, \bar{\nabla}_s^i\, U_{\mathbf{1}}^i\left(\boldsymbol{c}_{\mathbf{1}}^i(\boldsymbol{\lambda}^i)\right) \qquad k \in \mathbb{K},\ i \in \mathbb{I}.$$

Writing the first-order-condition this way, on changing the asset allocation $\boldsymbol{\lambda}_{\mathbf{1}}$ on $\boldsymbol{A}$ taking ratios

[1]Recall that agents are not endowed with assets so that changing prices does not change their income.

of any two components of the vector on the right hand side, gives the changes in the marginal rates of substitution between any two assets while the corresponding ratios on the left-hand-side gives the perceived changes of relative asset prices. Now suppose a competitive equilibrium is obtained in which this first-order-condition holds ignoring the $K$ Nash-terms. Then choosing the same portfolio as in the competitive equilibrium is also budget feasible in the situation with strategic interaction. Moreover, as prices are turned to your disadvantage, the perceived budget set in the case of strategic interaction is included in the budget set keeping prices as given. The first-order-condition shows that, moreover, the slope of the budget set anticipating your market impact coincides with that of the competitive budget set at those points where all agents choose the same portfolio. This is because at these points all $K$ Nash terms identical. Hence, also in the case of strategic behavior, independently of the risk aversion, the market outcome will be given by complete risk sharing.

— Please insert Figure 3 about here —

**CAPM and NoAggregateRisk** We illustrate this theorem by considering an economy without aggregate risk and two equally probable states $s = 1, 2$ in which two investors $i = 1, 2$ with identical wealth $w^1 = w^2$, compete for two assets $k = 1, 2$. Investors can act competitively or strategically. Asset 1 has payoff $(1, \alpha)$, while asset 2 has payoff $(0, 1 - \alpha)$ over states $1, 2$. The market structure is given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & \alpha \\ 0 & 1-\alpha \end{pmatrix}, \qquad 0 \leq \alpha \leq 1.$$

Note that this market has no aggregate risk, i.e. $\sum_k A_s^k = 1$ independent of $s$. The utility function $u^i := \mathbb{R}_+ \to \mathbb{R}$ considered is of the form $u^i(c) = c - \frac{\gamma}{2}c^2$. This function is identical across periods and also among consumers. Note that this function does not satisfy the INADA assumptions made above. Hence this "illustration" is not really covered by our previous theorems. Nevertheless, we see from Figure 4 that all implications of our theorems also hold for this important case.

— Please insert Figure 4 about here —

In order to study the case of **AGGREGATE RISK** consider the market $\boldsymbol{A}$ given as

$$\boldsymbol{A} = \begin{pmatrix} 2 & \alpha \\ 0 & 1-\alpha \end{pmatrix}, \qquad 0 \leq \alpha \leq 1,$$

while all other specifications are the same as in the example above, see Figure 4. One observation in this case is that consumers with identical characteristics $[\mathbb{U}^i, w^i]$ choose the same portfolio if market behavior among consumers is homogenous. Both for the economy in which both agents behave competitively and also for the case of strategic behavior the same portfolio is chosen. On the other hand if we consider an economy with identical consumer characteristics but with different market behavior, then in the presence of aggregate risk the portfolios differ.

The intuition for this observation is the following: The Nash equilibrium we have computed is a symmetric Nash equilibrium, i.e. a situation in which identical agents choose identical strategies. This symmetry is also true in the competitive equilibria. Moreover, the available total payoffs are independent of the market behavior we consider. Hence, since there are no redundant assets, with identical consumers‘ characteristics the portfolio choices in symmetric Nash equilibria coincide with those in the competitive equilibrium. But still competitive and Nash equilibria differ with respect to the money invested in the mutual fund. On the other hand, if we mix competitive with strategic behavior in one market, then the strategically acting agent will invest less in the assets and will consume more today so that he evaluates his portfolio of assets at a different second period wealth level. Hence, if relative risk aversion depends on the wealth level, as it does in the case of quadratic utilities, then both agents will choose different portfolios even though they have identical characteristics $[\mathbb{U}^i, w^i]$.

— Please insert Figure 5 about here —

This suggests that if the portfolio choice is independent from the wealth level, as it is in the case of constant relative risk aversion, then all investors should hold the same mutual fund. The next theorem states that even with aggregate risk 2pFS holds, if all investors have identical relative risk aversion.

**Theorem 3** *Suppose there are no redundant assets, i.e. rank* $\boldsymbol{A} = K$*. Moreover, assume all investors have identical second period relative risk aversion, i.e.* $u^i_{\mathbf{1}} = u_{\mathbf{1}}$ *for all* $i \in \mathbb{I}$*, where* $u_{\mathbf{1}} : \mathbb{R}_+ \to \mathbb{R}$ *is defined by*

$$u_{\mathbf{1}}(c) = \begin{cases} c^{\eta}, & 0 < \eta < 1 \\ \ln(c) & \end{cases} \qquad c > 0$$

*Then in every CNE 2pFS holds, i.e. there exists a common mutual fund* $\bar{\boldsymbol{\lambda}} \in \Delta^{K+1}_+$ *with* $\sum_{k=0}^K \bar{\boldsymbol{\lambda}}_k = 1$ *such that*

$$\hat{\boldsymbol{\lambda}}^i \in \left\langle \boldsymbol{\lambda}_0, \bar{\boldsymbol{\lambda}} \right\rangle \cap \Delta^{K+1}_+ \qquad \forall i \in \mathbb{I}.$$

*The mutual fund is of the form*

$$\bar{\boldsymbol{\lambda}}_k = 1/\mu \sum_s p_s \frac{A^k_s}{(\sum_k A^k_s)^{1-\eta}}, \tag{21}$$

*where* $\mu > 0$ *is a normalization constant so that* $\sum_{k=1}^K \bar{\boldsymbol{\lambda}}_k = 1$*.*

**Proof 4** *Part I: Recall that* $\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}^i\right)_{i\in\mathbb{I}}$ *is the vector of investment strategies on the market. Consider two investors* $i, j \in \mathbb{I}$ *with identical utility functions* $\mathbb{U}^i, \mathbb{U}^j : \mathbb{R}^{S+1}_+ \to \mathbb{R}$*. Note that* $\bar{\boldsymbol{\nabla}}^i U^i_{\mathbf{1}}(\boldsymbol{c}^i_{\mathbf{1}})$ *is homogenous of degree* $\nu \in \{-1, \eta - 1\}$ *for all* $i \in \mathbb{I}$*. Both 'see' the same price system* $\boldsymbol{q}$*. Hence, since rank* $\boldsymbol{A} = K$*, the associated linear map is injective and so the pre-image of* $\boldsymbol{q}$ *is unique. It thus follows that* $\bar{\boldsymbol{\nabla}}^i U^i_{\mathbf{1}}(\boldsymbol{c}^i_{\mathbf{1}}(\boldsymbol{\lambda}^i)) \bullet \mathcal{N}^i(\boldsymbol{\lambda}) = \bar{\boldsymbol{\nabla}}^j U^j_{\mathbf{1}}(\boldsymbol{c}^j_{\mathbf{1}}(\boldsymbol{\lambda}^j)) \bullet \mathcal{N}^j(\boldsymbol{\lambda})$ *and therefore* $\bar{\boldsymbol{\nabla}}^i U^i_{\mathbf{1}}(\boldsymbol{c}^i_{\mathbf{1}}(\boldsymbol{\lambda}^i)) \| \bar{\boldsymbol{\nabla}}^j U^j_{\mathbf{1}}(\boldsymbol{c}^j_{\mathbf{1}}(\boldsymbol{\lambda}^j))$*. Since* $\bar{\boldsymbol{\nabla}}^i U^i_{\mathbf{1}}(\boldsymbol{c}^i_{\mathbf{1}}(\boldsymbol{\lambda}^i))$ *and* $\bar{\boldsymbol{\nabla}}^j U^j_{\mathbf{1}}(\boldsymbol{c}^j(\boldsymbol{\lambda}^j))$ *are homogenous of the same degree* $\nu$ *and* $\boldsymbol{c}^i_{\mathbf{1}}, \boldsymbol{c}^j_{\mathbf{1}}$ *are homogenous of degree 1 in* $\boldsymbol{\lambda}^i, \boldsymbol{\lambda}^j$*, it follows that* $\boldsymbol{\lambda}^i \| \boldsymbol{\lambda}^j$ *for any pair* $i, j$*. Hence all investment strategies* $\{\boldsymbol{\lambda}^i\}$ *are co-linear and are in the same subspace, i.e. for every pair* $(i, j)$ *there exists a real valued scalar* $0 \leq \ell(i, j) \leq 1$ *such that* $\boldsymbol{c}^j = l(i, j)\boldsymbol{c}^i$*. Particularly if* $\hat{\boldsymbol{\lambda}}^i$ *is an CNE investment strategy, then there exists some factor* $\ell > 0$ *such that* $\bar{\boldsymbol{\lambda}} := \ell\hat{\boldsymbol{\lambda}}^i$ *and* $\sum_k \bar{\boldsymbol{\lambda}}_k = 1$ *is the unique mutual fund which spans the corresponding sum space* $\langle \boldsymbol{\lambda}_0, \bar{\boldsymbol{\lambda}} \rangle$*.*

*Part II: Recall that under 2pFS, with the notations from the proof of Theorem 2, the FOC reads*

$$\sum_s A^k_s \, \bar{\nabla}^i_s \, U^i_{\mathbf{1}} \left( \boldsymbol{c}^i_{\mathbf{1}}(\hat{\lambda}_0) \right) \left( 1 - \nu^i(\hat{\lambda}_0) w^i \right) \quad = \quad \bar{\boldsymbol{\lambda}}_k \sum_j (1 - \hat{\lambda}^j_0) w^j,$$

*where* $\boldsymbol{c}^i_s(\hat{\lambda}_0) = \left(\sum_k A^k_s\right) \nu^i(\lambda_0) w^i$ *is the consumption in state* $s$*. Assume that* $\hat{\lambda}^i_0 > 0$ *and define* $K(\hat{\lambda}^i_0) := \beta^i \left( \frac{\partial}{\partial c^i_{\mathbf{0}}} u^i_0(c^i_{\mathbf{0}}) \right)^{-1}$*. Note that* $K(\hat{\lambda}^i_0) \to 0$ *as* $\hat{\lambda}^i_0 \to 0$*, while it is positive and finite for*

$\lambda_0^i > 0$. *Then the FOC equivalently reads*

$$\sum_s A_s^k \, \bar{\nabla}_s^i \, U_{\mathbf{1}}^i \left( \boldsymbol{c}^i(\hat{\lambda}_0) \right) \left( 1 - \nu^i(\hat{\lambda}_0) w^i \right) = \bar{\boldsymbol{\lambda}}_k \frac{(1-\hat{\lambda}_0^i) w^i}{\nu^i(\hat{\lambda}_0^i) w^i} \quad (22)$$

$$\eta K(\hat{\lambda}_0^i) \left( \sum_s p_s A_s^k \left( \sum_k A_s^k \right)^{\eta - 1} \right) \left( 1 - \nu^i(\hat{\lambda}_0) w^i \right) (\nu^i(\hat{\lambda}_0) w^i)^{\eta - 1} = \bar{\boldsymbol{\lambda}}_k \frac{(1-\hat{\lambda}_0^i) w^i}{\nu^i(\hat{\lambda}_0^i) w^i} \quad (23)$$

$$\eta K(\hat{\lambda}_0^i) \left( \sum_s p_s \frac{A_s^k}{(\sum_k A_s^k)^{1-\eta}} \right) \left( 1 - \nu^i(\hat{\lambda}_0) w^i \right) (\nu^i(\hat{\lambda}_0) w^i)^{\eta} = \bar{\boldsymbol{\lambda}}_k (1-\hat{\lambda}_0^i) w^i, \quad (24)$$

*Inserting (24), we obtain*

$$\eta/\mu K(\hat{\lambda}_0^i) \left( 1 - \nu^i(\hat{\lambda}_0) w^i \right) (\nu^i(\hat{\lambda}_0) w^i)^{\eta} = (1-\hat{\lambda}_0^i) w^i \quad (25)$$

$$\eta/\mu \left( 1 - \nu^i(\hat{\lambda}_0) w^i \right) (\nu^i(\hat{\lambda}_0) w^i)^{\eta} = (1-\hat{\lambda}_0^i) w^i \, \beta^i \left( \frac{\partial}{\partial c_{\mathbf{0}}^i} u_0^i(c_{\mathbf{0}}^i) \right) \quad (26)$$

*The right-hand-side is strictly increasing in* $\lambda_0^i$ *and tends to* $0$ *if* $\lambda_0^i \to 1$, *while it tends to* $+\infty$ *if* $\lambda_0^i \to 0$. *To discuss the behavior of the right-hand-side, recall that* $\nu^i(\lambda_0^i) \to 0$ *if* $\lambda_0^i \to 1$, *while it remains positive and finite for* $\lambda_0^i < 1$. *Put* $x := \nu^i(\lambda_0^i) w^i$, *then* $x \in \mathbb{R}_+$. *The real valued function* $f(x) = (1-x)x^\eta$, $0 < \eta < 1$ *is concave,* $\frac{d}{dx} f(x)|_{x=0} = +\infty$, *and has two roots* $(0,1)$. *Hence there exist two solutions of equation (26), a trivial one* $\hat{\lambda}_0^i = 1$ *and a non trivial one* $0 < \hat{\lambda}_0^i < 1$.

Note that the general mutual fund, see equation 21, includes those for the case of NAR, log-utilities, and also of risk-neutrality: Indeed for $\eta = -1$ we get that $\bar{\boldsymbol{\lambda}}_k$ is the expected relative payoff of asset $k$, while for $\eta = 0$ we get that $\bar{\boldsymbol{\lambda}}_k$ is the relative expected payoff of asset $k$.

**Log utility function on a market with aggregate risk** For illustration we consider the same setting as defined above, i.e. a market with aggregate risk, but now consider the case that both investors have identical CRRA utility functions, particularly logarithmic ones, i.e. $u_{\mathbf{1}}^i(c) = \ln(c)$. The extended market structure thus is

$$\boldsymbol{A} = \begin{pmatrix} 2 & \alpha \\ 0 & 1-\alpha \end{pmatrix}, \qquad 0 \leq \alpha \leq 1.$$

For simplicity we assume that states 1 and 2 are equally probable, i.e. $p_1 = p_2 = 1/2$, and wealth distribution is $w^1 = w^2$.

— Please insert Figure 6 about here —

Theorem 3 states an interesting property of CNE. However it does not establish the existence of CNE with these properties. The next proposition shows that such an investment strategy in fact establishes a CNE.

**Proposition 1** *Let $\boldsymbol{\lambda}^i \in \langle \boldsymbol{\lambda}_0, \bar{\boldsymbol{\lambda}} \rangle$. Then there exists a real valued coefficient $0 \leq \lambda_0^i \leq 1$ such that*

$$\lambda^i = \lambda_0^i \boldsymbol{\lambda}_0 + (1 - \lambda_0^i)\bar{\boldsymbol{\lambda}}$$

*is a CNE investment strategy for investor $i$.*

**Proof 5** *Using notations as in the proof of Theorem 2 and by defining functions $\mathcal{F}$ and $\mathcal{G}_k$ by $\mathcal{F}(\lambda_0) := \boldsymbol{A}\bar{\boldsymbol{\nabla}}^i U_{\mathbf{1}}^i((\sum_k A_s^k)\nu^i(\lambda_0)w^i)$ and $\mathcal{G}_k(\lambda_0) := \bar{\boldsymbol{\lambda}}_k \sum_j (1 - \lambda_0^j)w^j$, the FOC reads*

$$\mathcal{F}_k(\lambda_0)(1 - \nu^i(\lambda_0)w^i) = \mathcal{G}_k(\lambda_0).$$

*Note that $0 < \mathcal{G}_k(\lambda_0) < \infty$ for any given $\lambda_0$ while if $\lambda_0^i \to 0$, then $\mathcal{F}_k(\lambda_0) \to 0$ and if $\lambda_0^i \to 1$ then $\mathcal{F}_k(\lambda_0) \to \infty$. Since both functions are continuous in $\lambda_0$, a solution exists.*

The next theorem shows that under 2pFS agents acting strategically invest less in the mutual fund than those acting competitively. As a consequence of this, the utility level of the agents in a market in which every agent behaves strategically is higher than the utility level in a competitive market. Note that this statement does not conflict with the first welfare theorem, i.e. with the Pareto-efficiency of competitive equilibria. From a central planning perspective, in our model the agents are strictly better off consuming almost all their wealth today and betting only very little on the asset market. This is because the assets are in fixed supply while the first period consumption good is in infinitely elastic supply.

**Theorem 4** *Let $\boldsymbol{\lambda}^{*i}(w^i) = \lambda_0^{i*}(w^i)\boldsymbol{\lambda}_0 + (1 - \lambda_0^*(w^i))\bar{\boldsymbol{\lambda}}$ be a CE. Then $\tilde{\boldsymbol{\lambda}}^i(w^i) = \tilde{\lambda}_0^i(w^i)\boldsymbol{\lambda}_0 + \left(1 - \tilde{\lambda}_0^i(w^i)\right)\bar{\boldsymbol{\lambda}}$ is a NE for some $\tilde{\lambda}_0^i(w^i) \geq \lambda_0^{i*}(w^i)$.*

**Proof 6** *Consider an economy with given wealth distribution $\boldsymbol{w} = (w^i, i = 1, ..., I)$ and assume that $\lambda(w^i) \in \left\langle \boldsymbol{\lambda}_0, \bar{\boldsymbol{\lambda}} \right\rangle$ is a CE. We show that there exists $\tilde{\lambda}_0^i$ such that $\tilde{\lambda}(w^i) = \tilde{\lambda}_0^i \boldsymbol{\lambda}_0 + (1 - \tilde{\lambda}_0^i)\bar{\boldsymbol{\lambda}}$ is a NE. For the sake of simplicity let $\lambda_0 = (\lambda_0^i) := \left(\lambda_0^i(w^i), i \in \mathbb{I}\right)$ be the vector of 0 period investments of agents $i$. Then define the following function $\mathcal{F}^i(\lambda_0) := \boldsymbol{A}\bar{\boldsymbol{\nabla}}^i U_{\mathbf{1}}^i(\boldsymbol{c}_{\mathbf{1}}^i(\lambda_0)) - \boldsymbol{q}^*$. In fact $\frac{\partial}{\partial \lambda_0^i}\mathcal{F}_k^i(\lambda_0) > 0$ since $\boldsymbol{c}_s^i(\lambda_0) \to 0$ if $\lambda_0^i \to 1$ and hence $\bar{\nabla}_s^i u_{\mathbf{1}}^i(\boldsymbol{c}_s^i) \to +\infty$ according to the INADA assumption on $u_{\mathbf{1}}^i$. The FOCs for CE then takes the form*

$$\mathcal{F}_k^i(\lambda_0) = 0.$$

*Let $\lambda_0^*$ such that for given $\boldsymbol{w} = (w^i)$, $\mathcal{F}_k^i(\lambda_0^*) = 0$ for all $k$. Finally define $\mathcal{G}^i(\lambda_0) := \boldsymbol{A}\bar{\boldsymbol{\nabla}}^i U_{\mathbf{1}}^i(\boldsymbol{c}_{\mathbf{1}}^i) \bullet \mathcal{N}^i(\lambda_0) - \boldsymbol{q}(\lambda_0)$. Let $\lambda_0^{\#}$ be such that $\boldsymbol{q}(\lambda_0^{\#}) = \boldsymbol{q}^*$. Then since $\mathcal{N}_k^i(\lambda_0^{\#}) \leq 1$, we have $\mathcal{G}_k^i(\lambda_0^{\#}) \leq \mathcal{F}_k^i(\lambda_0^{\#})$. Hence it follows that $\tilde{\lambda_0}$ implicitly defined by $\mathcal{G}^i(\tilde{\lambda_0}) = 0$ fulfills*

$$\tilde{\lambda_0} \geq \lambda_0^*,$$

*or equivalently $\tilde{\lambda_0}(w^i) \geq \lambda_0^*(w^i)$.*

Hence under two-period fund separation, thinking strategically, i.e. taking into account that prices "slip away" on increasing orders, does matter for the share of wealth invested in the mutual fund, however it does not affect the portfolio allocation within the group of assets.

# 6 Asset pricing implications

From Corollary 1 above it is clear that CE and NE prices are the same in the limit of infinitely large markets with homogenous investor's population, i.e. for $I \to \infty$. What about prices on markets in which some investors act strategically and others do not. The question is whether thinking strategically matters for asset prices on small markets. The next statement shows that relative asset prices are independent of the composition of market participants as long as two-period Fund Separation holds. Particularly if 2pFS holds, then relative asset prices in a pure competitive and a pure Nash investor population are identical to those in combined Competitive Nash economies.

**Corollary 4** *If 2pFS holds, relative prices are independent of the composition of the agent's population.*

**Proof 7** *According to the market clearing condition, under 2pFS prices fulfill $\hat{q}_k = \sum_{i\in\mathbb{I}} \hat{\lambda}_k^i w^i$ for all $k \in \mathbb{K}$. By Theorem 2 we have 2pFS in the CNE economy with the unique mutual fund $\bar{\boldsymbol{\lambda}}$. Hence we have for the prices of assets $k \in \mathbb{K}$*

$$\hat{q}_k = \bar{\boldsymbol{\lambda}}_k \sum_i (1 - \lambda_0^i) w^i$$

*such that $\frac{\hat{q}_k}{\hat{q}_j}$ is in fact independent of the partitioning of $\mathbb{I}$.*

Recall the two examples mentioned above. We in fact observe that for our respective conditions relative prices are identical in the different regimes. By $(C/C)$ we denote a regime in which both investors have competitive behavior, in a $(N/N)$ regime both investors behave strategically while in the $(C/N)$ regime investor 1 acts competitively while investor 2 behaves strategically. The following table gives the relatives prices on a market with aggregate risk and a market without, when both investors have CAPM preferences and follow different strategies. As above the market is

$$\boldsymbol{A} = \begin{pmatrix} 2 & \alpha \\ 0 & 1-\alpha \end{pmatrix}, \qquad 0 \le \alpha \le 1.$$

Note that, as mentioned above, the identity of prices in homogenous (C / C) and (N / N) economies is a result of the symmetry of the setting!

| $\alpha$ | **CAPM - NAR** (C/C) | (N/N) | (C/N) | **CAPM - AR** (C/C) | (N/N) | (C/N) |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1.33.. | 1.33.. | 1.226 |
| .1 | 1.22.. | 1.22.. | 1.22.. | 1.593 | 1.593 | 1.462 |
| .2 | 1.55.. | 1.55.. | 1.55.. | 1.917 | 1.917 | 1.755 |
| .3 | 1.857 | 1.857 | 1.857 | 2.33.. | 2.33.. | 2.1324 |
| .4 | 2.33.. | 2.33 | 2.33 | 2.889 | 2.889 | 2.634 |
| .5 | 2.99.. | 2.99 | 2.99 | 3.667 | 3.667 | 3.334 |
| .6 | 4.0 | 4.0 | 4.0 | 4.833.. | 4.833 | 4.383 |
| .7 | 5.66.. | 5.66 | 5.66 | 6.778 | 6.778 | 6.128 |
| .8 | 9.000 | 9.000 | 9.000 | 10.667 | 10.667 | 9.612 |
| .9 | 19.0 | 19.0 | 19.0 | 22.33.. | 22.33.. | 20.053 |

## 6.1 Derivatives

One field in finance which has taken market impact as a serious concern, is the field of derivatives in which slippage and liquidity holes have been taken into account when hedging a contingent claim.

A nice intuitive account of these effects for managing derivatives is given in Taleb [19], chapter 4. For a more rigorous analysis along these lines see Frey and Stremme [7] and Schönbucher and Willmot [18] who have adjusted the famous Black and Scholes formula for slippage of prices. This literature also recognizes that slippage has some upside: "Many large traders use their buying power to prop up the market in which they accumulate positions" Taleb [19], page 69. To the best of our knowledge, the pros and cons of the market impact have not been balanced systematically by this literature. Moreover, it is questionable to consider strategic interaction in which only one party is allowed to act strategically while the rest of the market is passive.

Introducing derivatives leads to a new strategic aspect of the model considered here. On changing demand for the underlying asset agents can change the payoffs of the derivative assets that are based on the prices of that underlying. Indeed in this case it turns out that even with logarithmic utility functions equilibria depend substantially on the form of market behavior!

We illustrate this aspect by the following simple model of a look-back option. The payoff matrix is given as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ \alpha & q_1 \end{pmatrix},$$

where $q_1$ is the price of asset 1 determined in the first period. I.e. the second asset pays the price of the first asset if state 2 occurs. Again states $s_1$ and $s_2$ are equally probable, both investors are identical, i.e. have the same endowment $w^1 = w^2$ and have the same logarithmic utility functions. Investors can act competitively or strategically. Hence there are three situations: Both act competitively, both act strategically, one investor acts competitively while the other investor acts strategically. The simulation, Figure 7 shows that the funds chosen by the investors differ significantly if both follow different strategies.

— Please insert Figure 7 about here —

# 7 Conclusions and Outlook

We have suggested a simple asset market model in which we analyzed competitive and strategic behavior simultaneously. We have shown that if for competitive behavior two-fund separation holds across periods then it also holds for strategic behavior. In this case the relative prices of the assets do not depend on whether agents behave strategically or competitively. Those agents acting strategically will however invest less in the common mutual fund. Constant relative risk aversion and absence of aggregate risk have been shown to be two alternative sufficient conditions for two-period fund separation. With derivatives further strategic aspects arise and strategic behavior is distinct from competitive behavior even for those utility functions leading to two-fund separation.

These results are first steps in building a new capital asset market model in which strategic interaction plays some role. Further research may endogeneize wealth by giving agents endowments in terms of assets. Moreover, the model should be extended to multiple periods.

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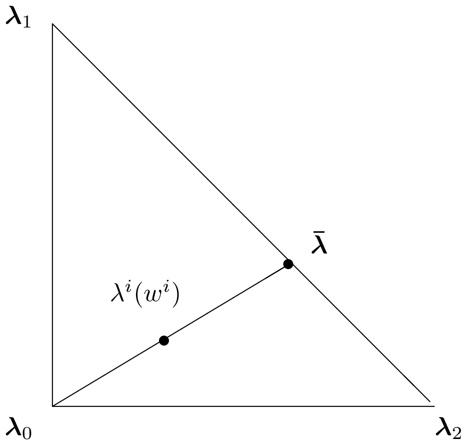


Figure 1: The simplex $\Delta^3_+$ of investment strategies $\boldsymbol{\lambda} = (\lambda_0, \boldsymbol{\lambda_1})$ over periods 0 and 1 on a market $\mathbf{A} \in \mathbb{R}^{2\times 2}_+$ displayed in $\mathbb{R}^2_+$.

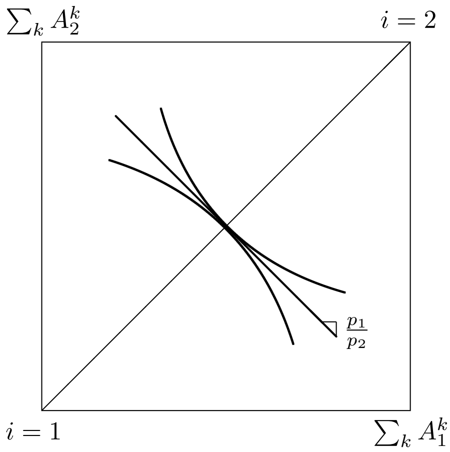


Figure 2: Complete Risk Sharing in Competitive and in Nash Equilibrium

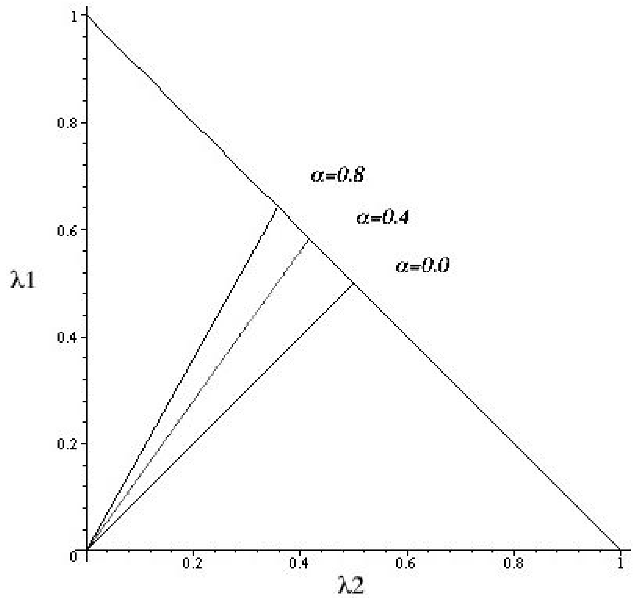


Figure 3: *Mutual funds for log utility functions on a market with aggregate risk depending on the market parameter* $\alpha$ *as defined in the examples.*

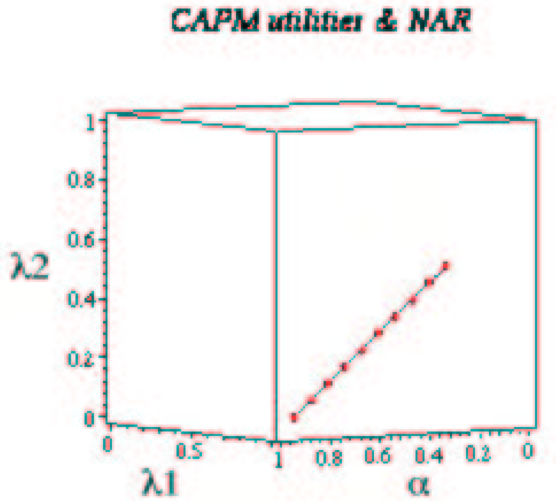


Figure 4: *Funds chosen by the two CAPM investors on a market WITHOUT aggregate risk. Funds of investors coincide if both have the same market behavior (dots). The solid line shows the common mutual fund chosen by BOTH investors even if they act according to different strategies, particularly investor* 1 *acts competitively and investor* 2 *acts strategically. This figure should be compared with the analogous setting for a market WITH aggregate risk*

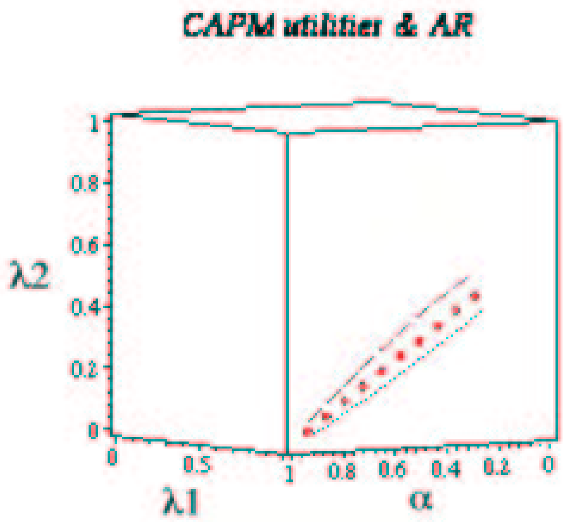


Figure 5: *Funds chosen by the two CAPM investors on the market WITH AGGRGATE RISK. Due to the symmetry of the situation funds of investors coincide if both have the same market behavior (dots), while they choose different funds on the asset market, displayed by lines (dashed for the competitive investor and dotted for the Nash investor), if they follow different strategies, i.e. one of them is acting strategically while the other behaviors competitively.*

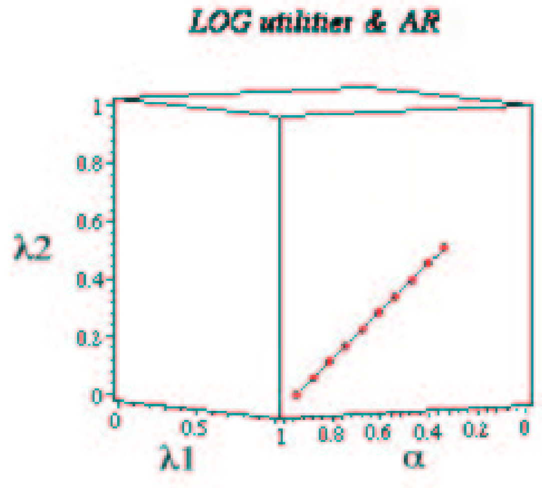


Figure 6: *Fund selection of investors with log utility functions on a market with aggregate risk. Dots represent mutual funds chosen if both investor follow the same strategy, while the line indicates mutual funds chosen if one acts strategically while the other competitively. Even if both investors follow different strategies they choose the same fund on the asset market.*

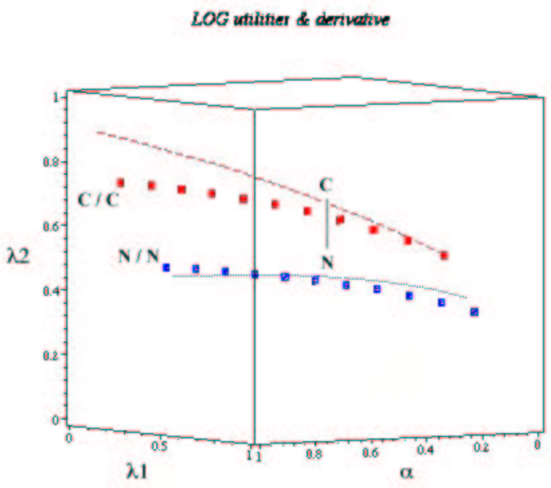


Figure 7: *Selection of funds in a small economy with derivatives. Because of symmetry both investors act identically in a C/C economy or in a N/N economy, while in a C/N economy both investors clearly behave differently.*